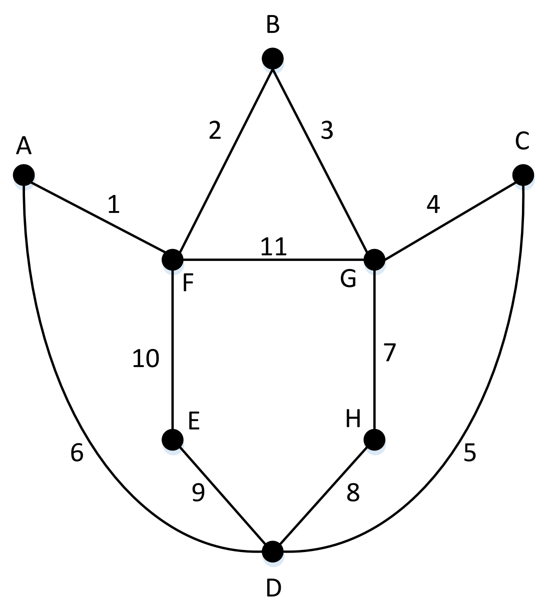
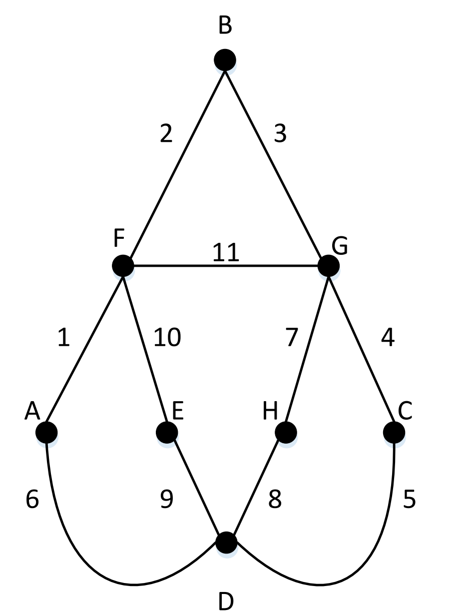
PART A – GRAPH THEORY – 30 MARKS

1. Equivalent Graphs (4 marks)

Label the vertices from A to H and edges from 1 to 11 of the graph on the right
to show that it is equivalent to the graph on the left.

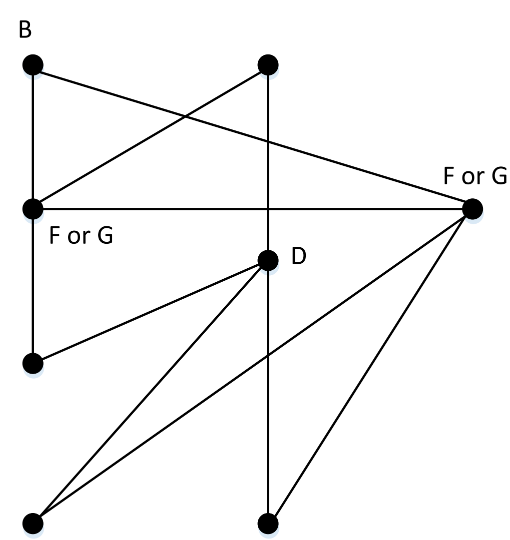
If we redraw the graph on the left slightly to line up vertices A, E, H, C, we get the graph on the right





You can see that the graph is symmetric along the axis formed by B and D. You can also see the the subgraph FAED is symmetric along the axix formed by F and D, and the subgraph GHCD is symmetric along the axis formed by G and D.

Therefore there will be four solutions, all of which with B and D in the same location, as shown on the diagram to the right.



2. Graph Degrees (6 marks)

For each of the following questions, either draw a graph with the requested properties, or explain **convincingly** (possibly by quoting a theorem) why such a graph cannot be drawn.

a) A graph with 5 vertices of degrees 5, 5, 4, 4, 3

The total degree of this graph would be 21. However, according to the handshake theorem, the total degree of a graph must be even. Therefore it is not possible to draw such a graph.

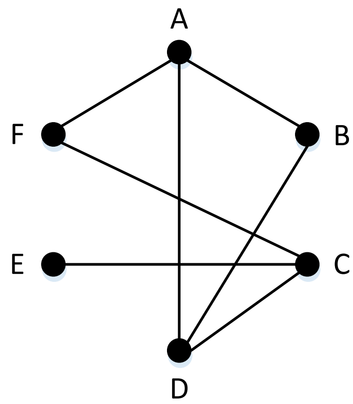
b) A graph with 5 vertices of degrees 5, 5, 4, 4, 4

This graph, on the other hand is possible because it has a total degree of 22 which is even. There are many different solutions.

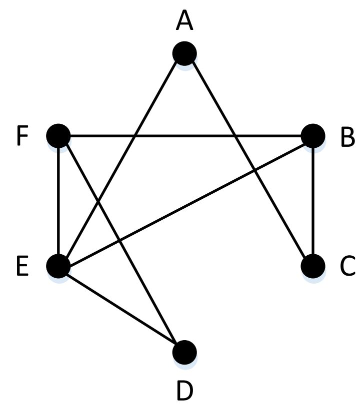
3. Acquaintance Graphs (8 marks)

Suppose that in a group of 6 people A, B, C, D, E, and F the following pairs of people are acquainted with each other: A and B, B and D, A and D, A and F, C and D, C and E, C and F.

a) Draw a graph G to represent who **is** acquainted with whom



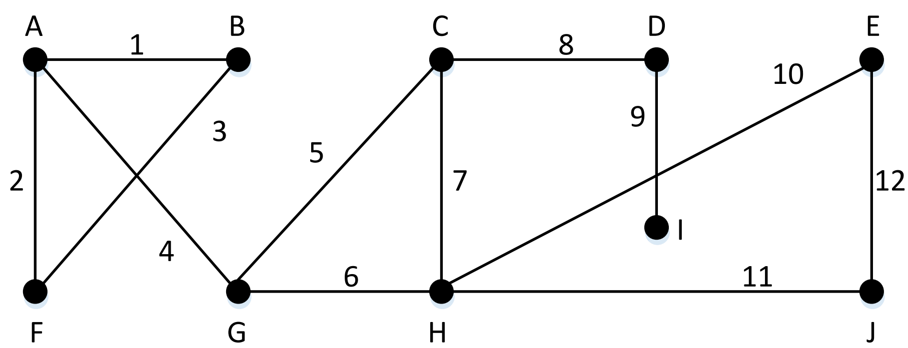
b) Draw a graph H to represent who **is** **not** acquainted with whom



c) What is G ∪ H? This is K_6, the complete graph with 6 vertices.

4. Connected Components (8 marks)

This question and the next one are based on this graph:



a) An edge of a graph whose removal disconnects the graph of which it is a part is called a “bridge”. List all the bridges of the graph above.

4, 8, 9

b) If you were to remove all the bridges that you listed in part a) from the graph, how many connected components would this graph have? List them.

There would be 4 connected components corresponding to the following sets of vertices {A,B,F}, {D}, {I}, and {C,G,H,E,J}

5. Walks (4 marks)

For each of the 4 walks in the graph in question 4, Indicate with True of False in the table below whether the walks in the graph in question 4 have each of the properties

Walk:	A1B3F2A4G	H6G5C7H11J12E10H	C8D8C7H6G5C	I9D8C5G4A
Path/Trail	T	T	F	T
(Simple) path	F	F	F	T
Closed walk	F	T	T	F
Circuit	F	T	F	F
Simple circuit	F	F	F	F

PART B – SEQUENCES AND RECURRENCE RELATIONS – 10 MARKS

Given the sequence a_n defined with the recurrence relation:

$a_1 = 1$

$a_n = a_{n-1} + n + 1$ for $n>1$

1. Terms of a Sequence (5 marks)

Calculate a_2, a_3, a_4, a_5, a_6
Keep your intermediate answers as you will need them in the next question.

$a_2 = a_1+2+1 = 1+2+1 = 4$

$a_3 = a_2+3+1 = 1+2+1+3+1 = 1+2+3+2 = 8$

$a_4 = a_3+4+1 = 1+2+1+3+1+4+1 = 1+2+3+4+3 = 13$

$a_5 = a_4+5+1 = 1+2+1+3+1+4+1+5+1 = 1+2+3+4+5+4 = 19$

$a_6 = a_5+6+1 = 1+2+1+3+1+4+1+5+1+6+1 \quad 1+2+3+4+5+6+5 = 264$

2. Iteration (5 marks)

Using iteration, solve the recurrence relation when $n\geq1$ (i.e. find an analytic formula for a_n). Simplify your answer as much as possible, showing your work. In particular, your final answer should not contain sums and products.

Based on the pattern above, when $n>1$

$$a_n = \sum_{i=1}^{n} i + n - 1 = \frac{n(n+1)}{2} + n - 1 = \frac{n(n+1)+2(n-1)}{2} = \frac{n^2+3n-2}{2}$$

substituting for $n=1$, the formula becomes $\frac{1^2+3-2}{2}$ = 2/2 = 1, so this formula also works when $n=1$

PART C – INDUCTION – 20 MARKS

Prove by induction that for all positive integers n, $\sum_{i=2}^{n} i(i-1) = \frac{n(n-1)(n+1)}{3}$
No other method is acceptable.
Be sure to lay out your proof clearly and correctly and to justify every step.

Let P(n) be the property $\sum_{i=2}^{n} i(i-1) = \frac{n(n-1)(n+1)}{3}$

We must prove $\forall n \in \mathbb{N}^+$ P(n)

Proof:

Basic Step: n=1

$\sum_{i=2}^{n} i(i-1) = \sum_{i=2}^{1} i(i-1) = 0$ because 1<2 so the sum has no terms

$\frac{n(n-1)(n+1)}{3} = \frac{1\times 0\times 2}{3} = 0$

Therefore P(1) is true

Inductive step:

Assume that P(k) is true for some positive integer k,

i.e , $\sum_{i=2}^{k} i(i-1) = \frac{k(k-1)(k+1)}{3}$ (inductive hypothesis)

We must show that P(k+1) is also true,

i.e. we must show that $\sum_{i=2}^{k+1} i(i-1) = \frac{(k+1)(k+1-1)(k+1+1)}{3}$

i.e. we must show that $\sum_{i=2}^{k+1} i(i-1) = \frac{(k+1)(k)(k+2)}{3}$

$\sum_{i=2}^{k+1} i(i-1)$

$= (k+1)k + \sum_{i=2}^{k} i(i-1)$ Taking the leading term out of the sum

$= (k+1)k + \frac{k(k-1)(k+1)}{3}$ by inductive hypothesis

$= k(k+1)(1+\frac{k-1}{3})$ factoring k(k+1)

$= \frac{k(k+1)(3+k-1)}{3}$ algebra

$= \frac{(k+1)(k)(k+2)}{3}$ algebra

QED